On the Stability Analysis of Rayleigh-Bénard Convection with Temperature Dependent Viscosity for General Boundary Conditions

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ABSTRACT

The problem of Rayleigh-Bénard convection with temperature dependent viscosity is investigated for all combinations of rigid and dynamically free boundaries. The principle of exchange of stabilities is shown to be valid for the problem which yields that the onset of convection is through stationary mode. Further, Rayleigh numbers for each possible combination of boundary conditions derived using Galerkin's technique. The effect of variable viscosity on the onset stationary convection for each case of boundary combination is computed numerically and depicted graphically. The present analysis yields that the positive values of temperature-dependent viscosity parameter has stabilizing effect on the onset of stationary convection while negative values have destabilizing effect. Further, various conclusions and results for the problem of Rayleigh-Bénard convection with constant viscosity for different cases of boundary conditions have been worked out and are discussed in detail.

Key words: thermal convection; principle of exchange of stabilities; Galerkin's technique; viscosity; boundary conditions; critical Rayleigh number

NOMENCLATURE

a^2	= Square of the wave number
a_c	= Critical wave number
T	= temperature
w	= Perturbation in vertical velocity
θ	= Perturbation in temperature
$R = \dfrac{g\alpha\beta d^4}{k\nu}$	= Rayleigh number
R_c	= Critical Rayleigh number
$p(= p_r + ip_i)$	= complex growth rate
$D \equiv \dfrac{d}{dz}$	= differentiation with respect to z
$f(T)$	= temperature-dependent viscosity
g	= acceleration due to gravity
d	= depth of the layer

Greek Symbols

α	= coefficient of volume expansion
β	= adverse temperature gradient
γ	= Coefficient of viscosity variation
k	= coefficient of thermometric conductivity
μ	= viscosity of the fluid
ρ	= density of the fluid
σ	= thermal Prandtl number
$\delta = \gamma\beta d$	= viscosity variation parameter
ν	= coefficient of kinematic viscosity

1. INTRODUCTION

The onset of thermal convection (thermal instability), in a fluid layer heated from below, generally known as Rayleigh-Bénard convection has been extensively studied by many researchers and is well suited to illustrate the many facets, mathematical and physical, of the general theory of hydrodynamic stability. The earliest experiments to demonstrate the onset of thermal instability in fluids in a definitive manner are those of Bénard [1]. The theoretical foundations for the correct interpretation of the results of the Bénard's experiments were laid by Lord Rayleigh [2]. Further remarkable contributions to the problem of Rayleigh-Bénard convection are due to Jeffrey's [3], Low [4], Pellew and Southwell [5] and Chandrasekhar [6].

It is well known fact that the viscosity is one of the properties of a fluid which are most sensitive to temperature ([7]). In the majority of the cases, viscosity becomes the only property which may have considerable effect on the heat transfer, whereas the temperature variation and dependence of other thermo-physical properties to temperature are often negligible. Torrance & Turcotte [8], Trompert and Hansen [9] and Booker and Stengel [10] observed that the viscosity of the liquids decreases with increasing temperature, while the reverse trend is observed in gases. Heat transfer and pressure drop characteristics are affected significantly with variations in the fluid viscosity. When the temperature is increased from 20 to 80°C, the viscosity of water decreases 2.7 times and the viscosity of air increases 1.4 times and the viscosity of engine oil decreases 24 times. Sherman [11] also reported that the viscosities of liquids vary considerably with temperature. Around room temperature (293 K), a 1% change in temperature produces a 7% change in the viscosity of water and approximately a 26% change in the viscosity of glycerol. Therefore, selection of the type of fluid and the range of operating temperatures are very important crucial parameters in the study of fluid dynamics.

The various experimental results related to the onset of convection of a fluid layer heated from below, one can summarize that; the convection takes place as the adverse temperature gradient reaches a critical value (dependent upon the depth and physical properties of the fluid including kinematic viscosity) and the motion tends towards a regular hexagonal pattern (cells). Experiments have shown that the direction of circulation, i.e. *ascent* or *descent* in the middle of the cell, changes from fluid to fluid. Generally, liquids and gases have opposite circulation; liquids having *ascent* and gases *descent* in the middle of the cell. Graham [12] has made an important suggestion that this different circulation may be a result of the fact that the kinematic viscosity varies with temperature in opposite manner in liquids and gases. Hence, this variation in viscosity behaviour with temperature may have a pronounced effect on the convective motions of the fluid. On the basis of above discussion, one can conclude that the range of operating temperatures and consequently the variation in viscosity of the fluid due to temperature is one of the major factor determining the direction of circulation and hence plays crucial role in controlling the dynamical behaviour of the fluid.

In most of the studies related to the Rayleigh-Bénard convection problems, the authors have considered the viscosity of the fluid as constant ([1–6]). However, many authors ([7–16]) have also investigated the onset of convection for the fluids with strongly temperature-dependent viscosity. Palm [13] studied the effect of a temperature dependent viscosity on the onset of Bénard problem. He explained theoretically from the observed results that the cells in steady convection approach a hexagonal form and the occurrence of *ascent* or *descent* in the middle of the cells depends on how the viscosity varies on the Rayleigh's result. Palm considered the particular case of a fluid layer between two free boundary surfaces with a not very realistic *cosine law* for the temperature dependent viscosity. Stengel *et al*. [14] compared Palm's results with these arising from the selection of an *exponential law*. The same problem was solved by Busse and Frick [15] for low viscous fluids with a *linear dependence* of the viscosity on temperature.

Critical review of the literature pertaining to the effect of temperature-dependent viscosity on the onset of convection reveals that the most of the studies are based on choice of T_0 (temperature of the upper/lower boundary) as the reference temperature. Nield [16] studied the effect of temperature-dependent viscosity on the onset of convection in a saturated porous medium and pointed out that study can also be carried out by using *average* temperature of the two boundaries as the reference temperature.

Although some more complex empirical expressions for viscosity variation $\mu(T)$ are also proposed in the literature, we shall discuss here only two laws, namely;

i) A *linear* temperature dependent viscosity variation law: $\mu(T) = \mu_0[1 - \gamma(T - T_0)]$. Here, μ_0 is the viscosity at lower boundary $z = 0$, and γ is a constant which takes positive values for liquids and negative for gases measures the viscosity variations with temperature and is given by ([17]);

$$\gamma = -\frac{d\mu}{\mu_0 dT} \text{ at } T = T_0. \tag{1}$$

ii) An *exponential* temperature dependent viscosity variation law: $\mu(T) = \mu_0 e^{-\gamma(T - T_0)}$. This relation is used for fluids with high viscosity.

On account of the generality and the simplicity the above discussed two laws, fit quite well for experimental data on restricted temperature ranges for a wide class of gases and liquids and involve only two parameters μ_0 and γ. Further, the linear viscosity law is the most commonly used relation in various studies and can be deduced from exponential law as a first order approximation for small viscosity variations. Banerjee *et al.* [18] presented a modified theory for classical Bénard convection problem, in which he demonstrated the role of specific heat variation due to temperature variation on the stability of the system and underlined the difference between the convection in hotter and cooler layer. He also suggested that, like other physical parameters, the viscosity of the fluid vary with temperature as; $\mu(T) = \mu_0[1 - \gamma(T - T_0)]$, where γ, is coefficient of viscosity variation and μ_0 and T_0 are the reference viscosity and temperature.

Nield [19] studied the stability of the horizontal layer of the fluid (with constant viscosity) when the thermal (concentration) gradient are not uniform for the different combinations of rigid and free boundary conditions, for linear as well as step-function profile of temperature. Recently, Sekhar and Jayalatha [20] have presented a comprehensive study of the Rayleigh-Bénard convection in viscoelastic liquids with temperature-dependent viscosity for different models of shear viscosity as a function of temperature for different combinations of boundary conditions. The authors concluded that stationary convection is the preferred mode of instability when the ratio of strain retardation parameter to stress relaxation parameter is greater than unity. For broader view and more details on the above subjects, refer to Nield and Bejan [21].

The main objective of the stability investigations in convective problems is to determine whether the onset of instability sets in either as a stationary convection or through oscillations and thus to find the numerical value of the critical Rayleigh number in each case. In most of the cases attention is focused on the determination of the stability criteria for the *ideal case* of both dynamically free boundaries, since in this case the *exact solution* of the problem can be obtained in a closed form leading to a dispersion relation. However, for the cases of physically *realistic* boundary conditions (i.e. when both boundaries are rigid or combinations of a rigid and a dynamically free boundary), it is difficult to obtain an exact solution in closed form, analytically. In such situations, the Rayleigh numbers are usually determined numerically ([6]).

The analysis of the present paper is primarily motivated by the above discussions and deals with the stability analysis of Rayleigh-Bénard convection with temperature dependent viscosity. The aim is to study analytically and numerically the effect of viscosity variation on the onset of Rayleigh-Bénard convection for each possible case of *realistic* as well as *idealistic* boundary conditions. The validity of the *principle of exchange of stabilities* (PES) is investigated analytically for this general problem. The values of the critical Rayleigh numbers for each combination of rigid and dynamically free boundary conditions are computed using the single term Galerkin expansion technique. This method is preferred over other such methods (e.g. shooting technique) to calculate the *critical eigenvalues* due to the less computational efforts involved. Further, the expansion functions used for unknown fields here, such as *polynomial* and *trigonometric* functions, are easy to evaluate and form a complete set of expansion functions (*i.e.* each function of the given space can be written as a linear combination of functions from the considered set), which are the essential properties that must be satisfied by expansion functions. The effects of variable viscosity and wave number on the onset of stationary convection is computed numerically and depicted graphically for each case of boundary combination, using the software *Mathematica*®. Various conclusions have also been presented based on the obtained results.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a Bénard layer of a viscous quasi-incompressible (Boussinesq) fluid confined between two horizontal boundaries $z = 0$ and $z = d$, maintained respectively at constant temperature T_0 and T_1 ($T_0 > T_1$). Following the usual steps of linear stability theory, the non-dimensional linearized perturbation equations for Rayleigh-Bénard convection problem with variable viscosity are given by ([6, 22]);

$$D\left[fD^3w + DfD^2w - 2a^2 fDw\right] + a^2\left(D^2 f + a^2 f\right)w - \frac{p}{\sigma}\left(D^2 - a^2\right)w = Ra^2\theta \tag{2}$$

$$\left(D^2 - a^2 - p\right)\theta = -w \tag{3}$$

together with the following cases of boundary conditions:

Case 1: *Both boundaries dynamically free*

$$w = 0 = \theta = D^2w \text{ at } z = 0 \text{ and } z = 1 \tag{4}$$

Case 2: *Both boundaries rigid*

$$w = 0 = \theta = Dw \text{ at } z = 0 \text{ and } z = 1 \tag{5}$$

Case 3: *Lower rigid and upper boundary free*

$$w = 0 = \theta = Dw \text{ at } z = 0 \text{ and } w = 0 = \theta = D^2w \text{ at } z = 1 \tag{6}$$

Case 4: *Lower free and upper boundary rigid*

$$w = 0 = \theta = D^2w \text{ at } z = 0 \text{ and } w = 0 = \theta = Dw \text{ at } z = 1 \tag{7}$$

The above equations have been made dimensionless by using the scale factors; $\frac{z}{d}, kd, d\frac{d}{dz}, \frac{nd^2}{k}, w, \frac{\theta k}{\beta d^2}, \frac{v_0}{k}$ for vertical distance, wave number, derivative D, pressure, vertical velocity, temperature and Prandtl number, respectively, where d is the characteristic length.

System of equations (2)–(3) together with either of the boundary conditions (4)–(7) constitutes an eigenvalue problem for R for given values of other parameters; namely σ, p and a. A given state of the system is stable, neutral or unstable according as p_r (real part of p) is negative, zero or positive respectively. Further, if $p_r = 0$ implies $p_i = 0$ for every wave number a, then the *principle of exchange of stability* (PES) is valid, which means that instability sets in as stationary convection, otherwise we shall have overstability at least when instability sets in as certain modes.

3. MATHEMATICAL ANALYSIS

3.1. Principle of exchange of stabilities

First of all, we shall investigate the validity of principle of exchange of stabilities for this general problem, using Pellew and Southwell [5] method.

Multiplying both sides of equation (2) by w^* (the complex conjugate of w), using the value of w^* from equation (3) in the right hand side of the resulting equation, we have

$$\begin{aligned}&\int_0^1 w^* D\left[fD^3w + DfD^2w - 2a^2 fDw\right]dz + a^2\int_0^1\left(D^2 f + a^2 f\right)\left|w\right|^2 + \frac{p}{\sigma}\int_0^1\left(\left|Dw\right|^2 + a^2w^2\right)dz\\ &= -Ra^2\int_0^1 \theta\left(D^2 - a^2 - p^*\right)\theta^*\, dz\end{aligned} \tag{8}$$

Integrating the various integrals of equation (8) by parts a suitable number of times over the vertical range of z and using either of the boundary conditions (4)–(7), we have

$$\begin{aligned}&\int_0^1 f\left[\left|D^2w\right|^2 + 2a^2\left|Dw\right|^2 + a^4\left|w\right|^2\right]dz + a^2\int_0^1 (D^2 f)\left|w\right|dz^2 + \frac{p}{\sigma}\int_0^1\left(\left|Dw\right|^2 + a^2\left|w\right|^2\right)dz =\\ &Ra^2\int_0^1 (\left|D\theta\right|^2 + a^2\left|\theta\right|^2 + p^*\left|\theta\right|^2)\,dz\end{aligned} \tag{9}$$

Comparing imaginary parts of equation (9), we get

$$p_i\left[\frac{1}{\sigma}\int_0^1\left(\left|Dw\right|^2+a^2\left|w\right|^2\right)dz+Ra^2\sigma\int_0^1\left|\theta\right|^2dz\right]=0 \tag{10}$$

Since, R is positive for the present configuration (fluid layer heated from below), the equation (10) clearly implies that $p_i = 0$, therefore, by the above definition, we conclude that PES is valid for the considered problem, and the result is uniformly valid for all cases of boundary combinations. This result further implies that the stationary convection is the only mode of onset of convection whether the viscosity is variable or constant (classical Bénard problem). It is remarkable to note here that the validity of PES leads to a notable mathematical simplification since the transition from stability to instability occurs via a marginal stationary state characterized by $p = 0$. Mathematically, this means that the marginally stable modes with $p_r = 0$ also have $p_i = 0$. Moreover, by setting $p = 0$, the original evolution of problem for the perturbations reduces to an eigenvalue problem for the Rayleigh number R.

3.2. The values of Rayleigh number

Since for the present problem the PES is valid, i.e. the instability sets in as stationary convection, therefore by putting $p = 0$, equations (2) and (3) reduce to the following forms;

$$D[fD^3w+DfD^2w-2a^2fDw]+a^2(D^2f+a^2f)w=Ra^2\theta \tag{11}$$

$$(D^2-a^2)\theta=-w \tag{12}$$

Hence, the system of equations (11)–(12) together with either of the boundary conditions (4)–(7) constitutes an eigenvalue problem for R for the onset of stationary convection.

We shall apply the Galerkin's method to find the critical value of Rayleigh numbers for each case of boundary combinations by taking a single term in the expansions for w and θ, following the analysis of Finlayson [23].

Writing the variables w and θ in equations (11)–(12) in the terms of the following trial functions;

$$w=Aw_1(z) \quad \text{and} \quad \theta=B\theta_1(z), \tag{13}$$

where, w_1 and θ_1 are the suitably chosen trial functions which satisfy the respective boundary conditions given in (4)–(7), and A and B are arbitrary constants.

Now, multiplying the resulting equations (obtained after substituting the above trial functions in (11) and (12)) by w_1 and θ_1 respectively, integrating each of the resulting equations by parts, using the relevant boundary conditions (4)–(7), we obtain the following equations;

$$A\int_0^1 f\left[(D^2w_1)^2+2a^2(Dw_1)^2+a^4(w_1)^2\right]dz+Aa^2\int_0^1(D^2f)(w_1)^2=Ra^2B\int_0^1\theta_1w_1dz \tag{14}$$

$$B\int_0^1\left[(D\theta_1)^2+a^2(\theta_1)^2\right]dz=A\int_0^1\theta_1w_1dz \tag{15}$$

Eliminating constants A and B from equations (14) and (15), we obtain following expression for Rayleigh number as;

$$R=\frac{\left\{\int_0^1[f\left[(D^2w)^2+2a^2(Dw)^2+a^4(w)^2\right]+a^2(D^2f)(w)^2]dz\right\}\left\{\int_0^1\left[(D\theta)^2+a^2(\theta)^2\right]dz\right\}}{a^2\left(\int_0^1w\theta dz\right)^2} \tag{16}$$

It is remarkable to note here that the expression (16) is valid for all cases of boundary conditions and for arbitrary function of viscosity variation $f(z)$.

We shall, now, find the expressions for Rayleigh numbers and consequently the values of the critical Rayleigh numbers R_c for four different cases of combination of boundary conditions.

The values of the Rayleigh numbers shall be respectively obtained for the *exponential* and *linear* non-dimensional viscosity variations given by;

$$f = e^{\delta z} \quad \text{and} \quad f = (1 + \delta z), \tag{17}$$

where, γ is the coefficient of viscosity variation and is related as; $\gamma \beta d = \delta$.

It is clear from expressions (17) that the viscosity variations are functions of magnitude of β (the temperature gradient across the layer), which means that for large temperature gradient across the layer the viscosity shall be smaller.

3.2.1. Case 1: Both Boundaries Dynamically Free

For the present case of boundary combination, we shall choose the following suitable polynomial trial functions satisfying the boundary conditions (4);

$$w = z^4 - 2z^3 + z \quad \text{and} \quad \theta = z(z-1) \tag{18}$$

Using trial functions (18) and viscosity variation laws (17) in expression (16) for R, we have the following expressions for R for *exponential* and *linear* case of viscosity variation, respectively;

$$\begin{aligned} R_{ff} = \frac{1}{289a^2\delta^9}\Big[& 11760(10+a^2)(144\delta^4(-12-6\delta-\delta^2+e^{\delta}(12-6\delta-\delta^2)\Big)+ \\ & a^4\Big(-20160-10080\delta-1440\delta^2+120\delta^3+48\delta^4-\delta^6+e^{\delta}(20160-10080\delta+1440\delta^2+120\delta^3 \\ & -48\delta^4+\delta^6))+2a^2\delta^2\Big(-15840-7920\delta-1152\delta^2+84\delta^3+36\delta^4-\delta^6-e^{\delta}\Big(15840-7920\delta \\ & +1152\delta^2+84\delta^3-36\delta^4+\delta^6\Big)\Big)\Big)\Big] \end{aligned} \tag{19}$$

$$\text{and} \quad R_{ff} = \frac{14(60480+30240\delta+a^2(18288+9144\delta)+a^4(1844+922\delta)+a^6(62+31\delta)}{867a^2}. \tag{20}$$

The minimum values $(R_{ff})_c$ for the present case of boundary conditions of Rayleigh number R_{ff} given in (19) for *exponential* variation of viscosity is given by;

$$\text{and} \quad (R_{ff})_c \approx 664.53 \text{ for } a^2 \approx 4.96 \text{ (as } \delta \to 0) \tag{21}$$

Further, for the case of *linear* variation of viscosity, R_{ff} attains minimum value $(R_{ff})_c$ as

$$(R_{ff})_c = 332.263\,(2+\delta) \text{ for } a^2 \approx 4.96, \tag{22}$$

where, $a^2 = 4.96$ is the only positive root of equation

$$31a^6 + 461a^4 - 15120 = 0. \tag{23}$$

It is clear from the above results that the values of critical Rayleigh number as obtained above for the case of *both dynamically free boundaries* is very close to the value 657.51 for $a^2 = 4.9348$, as obtained by Chandrasekhar [6] for the Rayleigh-Bénard convection with constant viscosity.

Table 1 shows the values of critical Rayleigh number for *linear* and *exponential* variation of viscosity Case 1 of boundary conditions. For the *exponential* variation, the values of $(R_{ff})_c$ are computed (using *Mathematica*$^{(R)}$) from expression (19) for different values of δ and the critical wave number a_c (the value of a at which the convection starts) for which R_{ff} attains its minima.
Figure 1 depicts the variation of Rayleigh number R_{ff} for different values of δ and a^2. The results obtained herein for the present case of boundary conditions are in good agreement with the existing results.

For *linear* variation, the values of $(R_{ff})_c$ are computed from expression (20) for different values of δ and for $a_c = 2.22$ at which R_{ff} attains its minima.

3.2.2. Case 2: Both Boundaries are Rigid

In this case of boundary conditions, the polynomial trial functions are given by

$$w = z^4 - 2z^3 + z^2 \text{ and } \theta = z(z-1) \tag{24}$$

Table 1. Variation of R_{ff} with δ for different values of a for Case 1 of boundary conditions

Linear variation			Exponential variation		
δ	**R_c**	**a_c**	**δ**	**R_c**	**a_c**
0.0	664.525	2.22	0.0001	664.562	2.22
0.1	697.76	2.22	0.1	697.78	2.1884
0.2	730.987	2.22	0.2	736.248	2.2284
0.3	764.214	2.22	0.3	776.413	2.225
0.4	797.441	2.22	0.4	819.78	2.223
0.5	830.667	2.22	0.5	866.63	2.2209
0.6	863.894	2.22	0.6	917.269	2.218
0.7	897.121	2.22	0.7	972.03	2.215
0.8	930.347	2.22	0.8	1031.28	2.212
0.9	963.574	2.22	0.9	1095.4	2.207

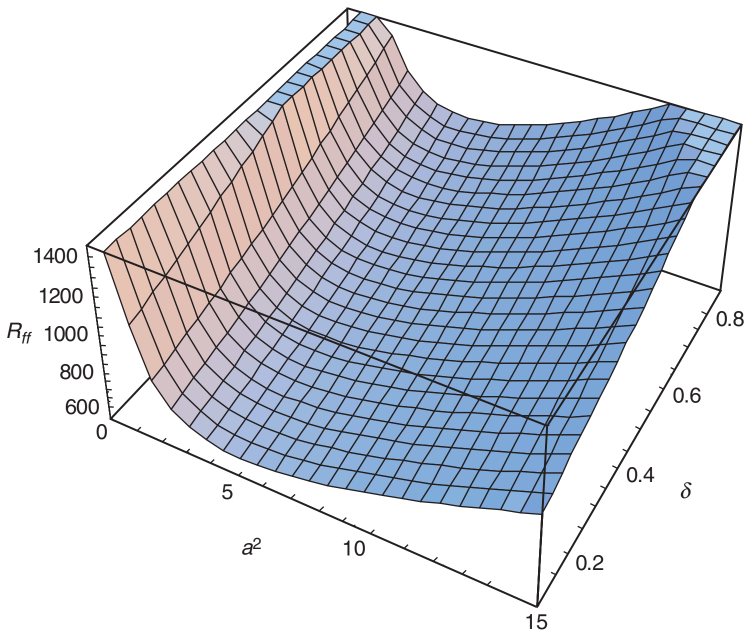

Figure 1. Variation of R_{ff} with a and δ.

Using trial functions (24) and viscosity variation laws (17) in expression (16) for R, we have the following expressions for R for *exponential* and *linear* case of viscosity variation, respectively;

$$R_{rr}=\frac{1}{3a^2\delta^9}\Big[7840\left(10+a^2\right)\Big(6a^4\Big(-1680-840\delta-180\delta^2-20\delta^3-\delta^4+e^\delta\left(1680-840\delta+180\delta^2\right.$$
$$\left.\left.-20\delta^3+\delta^4\right)\Big)+\delta^4\left(-864-432\delta-96\delta^2-12\delta^3-\delta^4+e^\delta\left(864-432\delta+96\delta^2-12\delta^3+\delta^4\right)\right)$$
$$+2a^2\delta^2\left(-7920-3960\delta-852\delta^2-96\delta^3-5\delta^4+e^\delta\left(7920-3960\delta+852\delta^2-96\delta^3+5\delta^4\right)\right)\Big)\Big] \tag{25}$$

$$\text{and}\quad R_{rr}=\frac{14\left[\left(2520\left(4+2\delta\right)+a^2\left(1488+744\delta\right)+a^4\left(68+34\delta\right)+a^6\left(2+\delta\right)\right)\right]}{27a^2} \tag{26}$$

The minimum values $(R_{rr})_c$ for the present case of boundary conditions of Rayleigh number R_{rr} given in (25) for *exponential* variation of viscosity is given by;

$$\left(R_{rr}\right)_c\approx 1750.063 \text{ at } a\approx 3.1165 \left(\text{at } \delta\to 0\right). \tag{27}$$

Further, for the case of *linear* variation of viscosity, R_{rr} attains the minimum values $(R_{rr})_c$

$$\left(R_{rr}\right)_c = 874.988(2+\delta) \text{ for } a_c = 3.1165 \tag{28}$$

where, a^2 is the only positive root of equation

$$a^6+17a^4-2520=0. \tag{29}$$

It is remarkable to point out that the obtained values of critical Rayleigh numbers for the case of *both rigid boundaries* is very close to the value 1749 at $a^2 = 9.71$, as obtained by Chandrasekhar [6] for the Rayleigh-Bénard convection with constant viscosity.

Table 2 shows the values of critical Rayleigh number for *linear* and *exponential* variation of viscosity Case 2 of boundary conditions. For the *exponential* variation, the values of $(R_{rr})_c$ are computed from expression (25) for different values of δ and a_c at which R_{rr} attains its minima. Figure 2 depicts the variation of Rayleigh number R_{rr} for different values of δ and a^2. The results obtained herein for the present case of boundary conditions are in good agreement with the earlier.

For *linear* variation, the values of $(R_{rr})_c$ are computed from expression (26) for different values of δ and for $a_c = 3.1165$ at which R_{rr} attains its minima.

Table 2. Variation of R_{rr} with δ for different values of a for Case 2 of boundary conditions

Linear variation			**Exponential variation**		
δ	R_c	a_c	**δ**	R_c	a_c
0.0	1749.97	3.1165	0.0001	1750.06	3.1165
0.2	1837.47	3.1165	0.2	1938.65	3.1052
0.3	1924.97	3.1165	0.3	2044.96	3.1139
0.4	2012.47	3.1165	0.4	2159.48	3.1162
0.5	2099.97	3.1165	0.5	2283.28	3.1218
0.6	2187.47	3.1165	0.6	2417.21	3.122
0.7	2274.97	3.1165	0.7	2562.16	3.124
0.8	2362.47	3.1165	0.8	2719.12	3.126
0.9	2449.97	3.1165	0.9	2889.17	3.0656

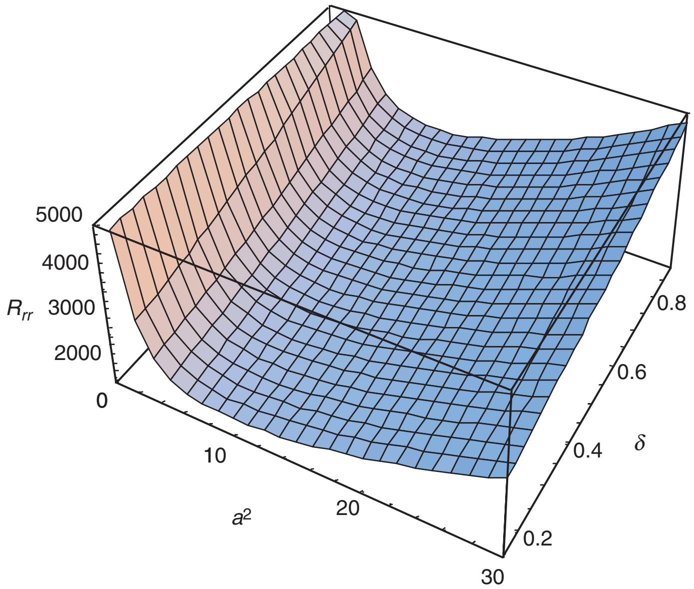

Figure 2. Variation of R_{rr} with a and δ.

3.2.3. Case 3: Lower Rigid and Upper Free Boundaries

In this case of boundary conditions, the polynomial trial functions are given by

$$w = 2z^4 - 5z^3 + 3z^2 \text{ and } \theta = z(z-1) \tag{30}$$

Using trial functions (30) and viscosity variation laws (17) in expression (16) for R, we have the following expressions for R for *exponential* and *linear* case of viscosity variation, respectively;

$$\begin{aligned} R_{rf} = \frac{1}{169a^2\delta^9}\Big[& 11760\left(10+a^2\right)\left(-18\delta^4\left(384+240\delta+66\delta^2+10\delta^3+\delta^4-6e^{\delta}\left(64-24\delta+3\delta^2\right)\right)\right. \\ & +a^4\left(-36\left(2240+1400\delta+370\delta^2+50\delta^3+3\delta^4\right)+e^{\delta}\left(80640-30240\delta+3240\delta^2+240\delta^3\right.\right. \\ & \left.\left.-72\delta^4+\delta^6\right)\right)+2a^2\delta^2\left(-18\left(3520+2200\delta+584\delta^2+80\delta^3+5\delta^4+e^{\delta}\left(63360-23760\delta\right.\right.\right. \\ & \left.\left.\left.\left.+2592\delta^2+168\delta^3-54\delta^4+\delta^6\right)\right)\right)\right)\Big] \end{aligned} \tag{31}$$

$$\text{and} \quad R_{rf} = \frac{7\left(10+a^2\right)\left[216a^2\left(8+5\delta\right)+1512\left(12+5\delta\right)+a^4\left(76+43\delta\right)\right]}{507a^2} \tag{32}$$

The minimum values $(R_{rf})_c$ for the present case of boundary conditions of Rayleigh number R_{rf} given in (31) for *exponential* variation of viscosity is given by;

$$\left(R_{rf}\right)_c \approx 1138.28 \text{ at } a \approx 2.66967 \left(\text{as } \delta \to 0\right). \tag{33}$$

Further, for the case of *linear* variation of viscosity, R_{rf} given in (32) attains its minimum values $(R_{rf})_c$ for different values of a^2, given by the positive root of equation

$$\left(76+43\delta\right)4a^6+\left(1244+755\delta\right)a^4-\left(90720+37800\delta\right)=0 \tag{34}$$

It is to note here that the value of $(R_{rf})_c$ for the case of *lower rigid and upper free boundaries* is very close to the value 1138.7 at a = 2.66967 as obtained by Chandrasekhar [6] for the Rayleigh-Bénard convection with constant viscosity.

Table 3 shows the values of critical Rayleigh number for *linear* and *exponential* variation of viscosity for Case 3 of boundary conditions.

For the *exponential* variation, the values of $(R_{rf})_c$ are computed from expression (31) for different values of δ and a_c at which R_{rf} attains its minima. Figure 3 depicts the variation of Rayleigh number R_{rf} for different values of δ and a^2. The results obtained herein for the present case of boundary conditions are in good agreement with the existing results.

For *linear* variation, the values of $(R_{rf})_c$ are computed from expression (32) for different values of a_c for various given values of δ. It is to point out here that we cannot determine a^2 explicitly, because the coefficients of equation (34) is dependent of δ whereas in the foregoing cases, equations (23) and (29) were independent of δ.

3.2.4. Case 4: Lower Free and Upper Rigid Boundaries

These types of boundary conditions are satisfied by the polynomial trial functions

$$w = 2z^4 - 3z^3 + z \text{ and } \theta = z(z-1) \tag{35}$$

Table 3. Variation of R_{ff} with δ for different values of a for Case 3 of boundary conditions

Linear variation			Exponential variation		
δ	**R_c**	**a_c**	**δ**	**R_c**	**a_c**
0.0	1138.7	2.6698	0.0001	1138.28	2.66
0.2	1254.35	2.6511	0.2001	1263.46	2.638
0.3	1312.12	2.64301	0.3	1333.55	2.637
0.4	1369.86	2.63562	0.4	1409.08	2.628
0.5	1427.57	2.62883	0.5	1490.66	2.617
0.6	1485.25	2.62258	0.6	1578.83	2.606
0.7	1542.91	2.6168	0.7	1674.18	2.595
0.8	1600.55	2.61144	0.8	1777.33	2.585
0.9	1558.18	2.60646	0.9	1888.98	2.573

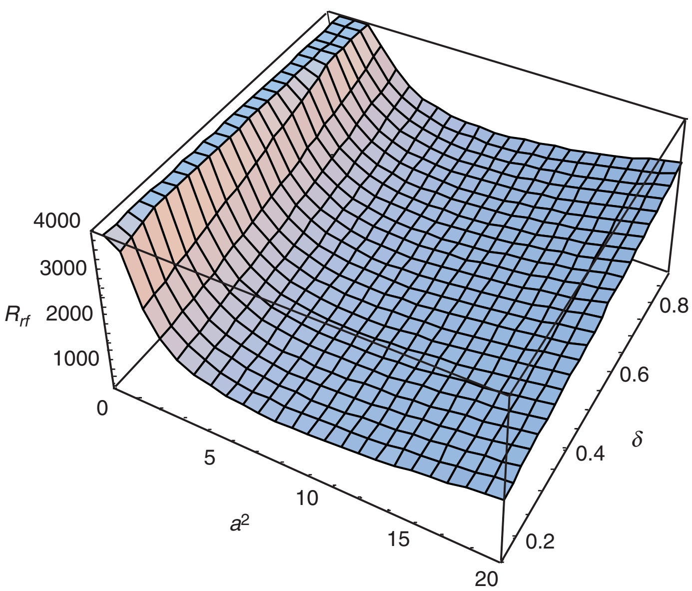

Figure 3. Variation of R_{rf} with a and δ.

Using trial functions (35) and viscosity variation laws (17) in expression (16) for R, we have the following expressions for R for *exponential* and *linear* case of viscosity variation, respectively;

$$\begin{aligned} R_{fr} = \frac{1}{169a^2\delta^9}\Big[& 11760\left(10+a^2\right)\Big(-18\delta^4\left(-6\left(64+24\delta+3\delta^2\right)+e^{\delta}\left(384-240\delta+66\delta^2-10\delta^3+\delta^4\right)\right) \\ & +a^4\Big(\left(80640+302400\delta+3240\delta^2-240\delta^3-72\delta^4+\delta^6\right)-36e^{\delta}\big(2240-1400\delta \\ & +370\delta^2-50\delta^3+3\delta^4\big)\Big)+2a^2\delta^2\Big(\big(63360+237060\delta+2592\delta^2-168\delta^3-54\delta^4+\delta^6 \\ & -18e^{\delta}\left(3520-2200\delta+584\delta^2-80\delta^3+5\delta^4\right)\big)\Big)\Big)\Big] \end{aligned} \tag{36}$$

$$\text{and}\quad R_{fr} = \frac{7\left(10+a^2\right)\left[216a^2\left(8+3\delta\right)+1512\left(12+7\delta\right)+a^4\left(76+33\delta\right)\right]}{507a^2} \tag{37}$$

The minimum values $(R_{fr})_c$ for the present case of boundary conditions of Rayleigh number R_{fr} given in (36) for *exponential* variation of viscosity is given by;

$$\left(R_{fr}\right)_c \approx 1138.28 \text{ at } a \approx 2.66967 \left(\text{as } \delta \to 0\right). \tag{38}$$

Further, for the case of *linear* variation of viscosity, R_{fr} given in (37) attains its minimum values $(R_{fr})_c$ corresponding to the positive root of the equation

$$\left(76+43\delta\right)4a^6+\left(1244+755\delta\right)a^4-\left(90720+37800\delta\right)=0 \tag{39}$$

Moreover, it is clear from the above results that the value of $(R_{fr})_c$ for the case of *lower free and upper rigid boundaries* is very close to the value 1100.65 at a = 2.682, as obtained by Chandrasekhar [6] for the Rayleigh- Bénard convection with constant viscosity.

Table 4 shows the values of critical Rayleigh number for *linear* and *exponential* variation of viscosity for Case 4 of boundary conditions.

For the *exponential* variation, the values of $(R_{fr})_c$ are from expression (36) for different values of δ and a_c at which R_{rf} attains its minima. Figure 4 depicts the variation of Rayleigh number R_{fr} for different values of δ and a^2. The results obtained herein for the present case of boundary conditions are in good agreement with the existing results.

For *linear* variation, the values of $(R_{fr})_c$ are computed from expression (37) for different values of a_c for various given values of δ. In this case also, we cannot determine a^2 explicitly, because the coefficients of equation (39) are dependent of δ.

Table 4. Variation of R_{rr} with δ for different values of a for Case 4 of boundary conditions

Linear variation			Exponential variation		
δ	R_c	a_c	δ	R_c	a_c
0.0	1138.7	2.6698	0.0001	1138.76	2.66978
0.2	1250.61	2.68856	0.2	1259.48	2.68756
0.3	1306.51	2.69677	0.3	1327.03	2.6999
0.4	1362.38	2.70432	0.4	1399.91	2.71038
0.5	1418.22	2.71112	0.5	1478.58	2.72045
0.6	1474.04	2.7177	0.6	1563.56	2.73027
0.7	1529.83	2.7237	0.7	1655.38	2.74004
0.8	1585.6	2.72928	0.8	1754.66	2.7499
0.9	1641.36	2.73447	0.9	1862.06	2.7594

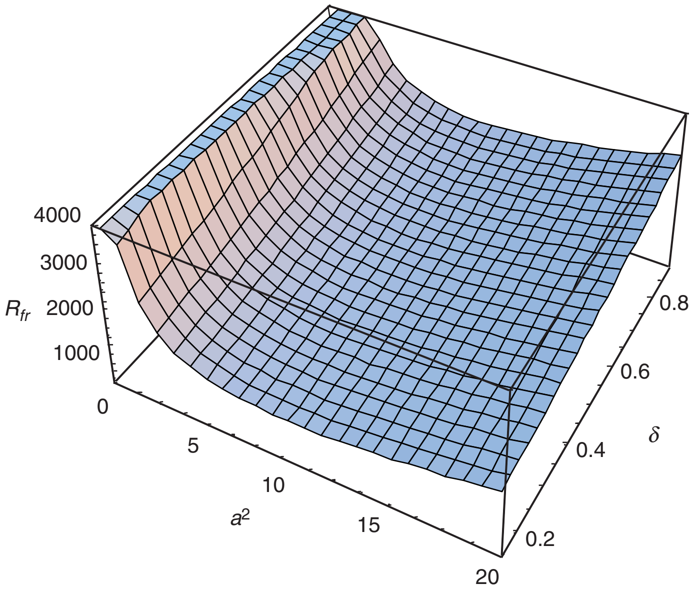

Figure 4. Variation of R_{fr} with a and δ.

4. NUMERICAL RESULTS AND DISCUSSIONS

We shall now present the various numerical results regarding the variation of Rayleigh numbers with respect to wave number and temperature-dependent viscosity in the above tables and 3-dimensional graphs. The values of Rayleigh numbers for different cases of boundary conditions is obtained by using Galerkin's method. As discussed earlier, this procedure not only gives reasonable result with minimum mathematical computation but has the advantage of keeping track of the effects, arising due to variations in the parameters. The variation of critical Rayleigh numbers R_c, for each case of boundary conditions, with respect of δ and a^2 are computed using *Mathematica* $^{(R)}$ and are presented in Tables 1–4 below.

The variation of Rayleigh numbers with respect of δ and a^2 for each case of boundary conditions for the exponential law of viscosity variation given in (17) is plotted graphically by using *Mathematica* $^{(R)}$. Figures 1–4, depict this variation of Rayleigh numbers with respect of δ and a^2, given by expressions (19), (25), (31) and (36) respectively for Case 1, Case 2, Case 3 and Case 4 of boundary conditions.

From the obtained results, one can easily deduce that in the limiting case of classical fluids (when the viscosity is taken to be a constant, i.e. $\delta = 0$) the results obtained for the present problem for each case of boundary conditions are in close agreement with the results obtained by Chandrasekhar [6] for the respective cases of boundary conditions. One can also observe from the above analysis that for the Case 1 and Case 2 of boundary conditions, the critical wave number a_c is independent of δ for *linear* variation of temperature-dependent viscosity and thus the critical Rayleigh numbers for these cases of boundary conditions are respectively given by;

$$\left(R_{ff}\right)_c = 332.263(2+\delta) \text{ for } a_c = 2.22697 \tag{40}$$

$$\text{and} \left(R_{rr}\right)_c = 874.988(2+\delta) \text{ for } a_c = 3.1165, \tag{41}$$

which are linear function of δ and can be easily computed.

However, for the case of *exponential* variation of temperature-dependent viscosity, a_c is depending upon δ and hence the critical Rayleigh numbers too. When $\delta \rightarrow 0$, the results obtained for *linear* and *exponential* variation of temperature-dependent viscosity have the same values of Rayleigh numbers and tend to the same values as obtained by Chandrasekhar [6] for the case of constant viscosity. Hence, the results obtained are general in nature and are in close agreement with the existing numerical results.

5. CONCLUSIONS AND FUTURE RECOMMENDATION

The present paper deals with the problem of thermal convection with temperature-dependent viscosity with the different combinations of rigid and dynamically free boundaries and carries out the stability analysis of eigenvalue problem for Rayleigh-Bénard convection with temperature dependent viscosity. The resulting eigenvalue problem for Rayleigh-Bénard convection with temperature dependent viscosity contains variable coefficients contrary to the classical case wherein viscosity is constant, which introduces extra complexities, in mathematical point of view. The *principle of exchange of stability* (PES) is shown to be valid for this general problem by following the Pellew and Southwell's [5] technique, which yields that the onset of convection, is through stationary mode.

It is clear from the obtained results that the temperature-dependent viscosity has stabilizing effect on the onset of stationary convection for the positive values δ. Further, for the case of negative δ, one can easily see by following the present analysis that it has destabilizing effect on the onset of stationary convection. It is also evident from these results that the *exponential* variation of temperature-dependent viscosity has more stabilizing effect than the *linear* variation of viscosity in each case of boundary conditions.

The present analysis can also be easily carried out for the convective instability problems under the varying assumptions of hydrodynamics and hydromagnetics and also for Non-Newtonian fluids. The analogous effect of various other temperature-dependent viscosity laws (based on the viscosity at the mean of the boundary temperatures as suggested by [14]) and temperature-dependent parameters on the onset of convection can also be investigated following the present analysis.

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